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A causal statistical family of dissipative divergence-type fluids

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Abstract. We define a particular class of dissipative relativistic fluid theories of divergence type with a statistical origin, in the sense that the three tensor fields appearing in the theory can be expressed as the first three moments of a suitable distribution function. In this set of theories the causality condition for the resulting system of hyperbolic partial differential equations is very simple and allows one to identify a subclass of manifestly causal theories, which are so for all states outside equilibrium for which the theory preserves this statistical interpretation condition. This subclass includes the usual equilibrium distributions, namely Boltzmann, Bose or Fermi distributions, according to the statistics used, suitably generalized outside equilibrium. Therefore, this gives a simple proof that they are causal in a neighbourhood of equilibrium. Unfortunately, these theories cannot retain their statistical character over the whole manifold of non-equilibrium states. Indeed, as we shall show, they cannot even be defined in any whole neighbourhood containing the equilibrium submanifold. This fact leads us to speculate that a possible origin of this behaviour is an inconsistency between the 14-parameter Grad truncation and the requirement of the existence of an entropy law. We also define a particular class of dissipative divergence-type theories with only a pseudostatistical origin. Some elements of this class do not have the previous inconsistency between the 14-parameter Grad truncation and the requirement of the existence of an entropy law. The set of pseudostatistical theories also contains a subclass (including the one already mentioned) of manifestly causal theories.

1. Introduction

In the last few years there has been a considerable effort to understand dissipation in relativistic theories of fluids. Straightforward generalizations to relativity of the Navier–Stokes scheme [1] resulted in systems with an ill-posed initial value formulation [2]. Even if these generalizations would have worked, they would have yielded a parabolic system, well-posed in the mathematical sense, but unacceptable on physical grounds due to the presence of infinite propagation velocities. Thus, alternative theories were proposed resulting in a formalism having an extended number of dynamical variables and where dissipation at the microscopic level is completely different to the standard parabolic dissipation of Navier–Stokes equations but which behaves, at measurable scales and in most circumstances‡, in the same way [4, 5]. Due to the problems encountered on the generalizations alluded above, one of the basic requirements to be checked on these alternative theories was their relativistic causality [6, 7].

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‡ There are experimental situations where this is not the case, see for instance, [3] and references therein.

Some of these dissipative fluid theories can be derived from certain approximations in kinetic theory. These approximations or truncations are based on the expectation that the dynamics of the Boltzmann equation on the cotangent space can be well approximated by the dynamics of a finite number of spacetime fields (macroscopic variables), satisfying certain partial differential equations. These macroscopic systems of equations are obtained through appropriate integrals over all possible microscopic particle momenta of the transport equation for the distribution function. The equations for the different moments have always the same structure: they equate total divergences of a given moment with its associate source of dissipation [8]. The order of approximation desired depends on the number of macroscopic variables (the order of truncation) considered (that is, the number of moments for the macroscopic equations). For example, the lowest-order approximation, which gives perfect fluid theories, is obtained by considering that the distribution function depends on five macroscopic variables grouped into two scalars, a spacetime scalar field and the contraction of a spacetime vector field with the microscopic particles momentum, while the five macroscopic equations are the equations for the first two moments of the distribution function. The next level of approximation is called the 14-parameter Grad truncation [8–10]. The expectation is to obtain more detailed information from the Boltzmann dynamics of certain distribution functions than the one provided by the perfect fluid description, with the assumption that these distribution functions depend on nine extra macroscopic variables. These extra variables are related to dissipative processes in the fluid and are given as components of a two rank trace free symmetric tensor field. The dependence of the distribution function on this tensor is through the contraction with two microscopic momenta. We now obtain a consistent system of equations when we consider the equations for the first three moments, but now the physical interpretation of the third moment is not clear.

We study here the truncations in kinetic theory in the context of the divergence-type fluid theories [6, 11, 12] (see appendix A). Therefore, besides the dynamical fluid equations, we to assume a *dynamical entropy law*, that is, the existence of an extra vector field in the theory with two properties. The first and most restrictive, is that its divergence by the sole virtue of the above equations, is a function of the basic fields and not of any of its derivatives; the second and less restrictive, is that this function is non-negative. Physically this means that there exists on the spacetime manifold, through Gauss's Theorem, a non-decreasing function along time slices. The first condition in the dynamical entropy law imposes severe restrictions in the theory, in particular it implies the existence of a specific set of dynamical fluid variables and a single scalar function of these variables called the *generating function*. This generating function completely determines the principal part of the system of dynamical equations—in the sense of the theory of partial differential equations—while for these particular variables the dynamical equations are explicitly symmetric. These facts are important in studying the causal properties of the corresponding fluid dynamics.

In section 2 we first define a subclass of non-dissipative divergence-type theories that we call *statistical*. The motivations for introducing these theories are given in appendix B. These perfect fluid theories have a statistical origin in the sense that the particle-number current and the stress-energy tensor can be expressed as the first two moments of some distribution function. This requirement singles out a generating function for the perfect fluid theory as a functional of the corresponding distribution function. We obtain a very simple condition on the associated distribution function, which ensures causality of the whole perfect fluid theory. We call the theories satisfying this condition *manifestly causal theories*. They include as particular cases the Boltzmann, Bose, and Fermi gases.

From the statistical non-dissipative divergence theory we can easily define a *dissipative* divergence theory while keeping its statistical origin. We shall say in this case that the dissipative theory thus obtained is a *natural extension* of the non-dissipative one. As in the perfect fluid case, the requirement of a statistical origin in the fluid theory singles out a generating function for the dissipative fluid theory as a functional of the corresponding distribution function which now belongs to the 14-parameter Grad truncation class of distribution functions. This extension has an important property. If the original non-dissipative theory is manifestly causal, then its dissipative extension is also manifestly causal. In particular, since the equilibrium theories corresponding to the Boltzmann, Bose and Fermi gases can be cast in the form of manifestly causal non-dissipative theories, we conclude that their natural extensions are also manifestly causal, thus considerably generalizing and simplifying the works of [12–15]. In spite of its naturality, this extension has some drawbacks, as is explained in section 2 that it is only well defined for some values of the dissipation variables, which, in particular, do not cover any whole neighbourhood of the local equilibrium submanifold. Thus, some compatible, but otherwise arbitrary, extension has to be made in order to allow the generating function to be well defined for the remaining values that the dissipation variables can take.

Later on, in section 3, we relax the statistical condition and define theories with a *pseudostatistical* origin, in the sense that the equations for the particle number and the stress-energy tensor can be thought of as the equations for the moments of different, although related, distribution functions. In this class of pseudostatistical theories there are cases where the theory is well defined in a whole neighbourhood of equilibrium; therefore, these theories do not have the drawbacks of statistical ones. Again we shall find a subclass of manifestly causal pseudostatistical theories which includes the former manifestly causal statistical theories, and again we show that manifestly causal non-dissipative theories have natural dissipative extensions which are also manifestly causal.

In section 4 we discuss the expression for the entropy of these theories. The requirement of an entropy law is so strong that even singles out a unique form for the entropy as a functional of the generating function. The entropy we deal with here is a *dynamical entropy* in the sense that it appears as further relations (equations) between the fields as a consequence of its dynamics. We find that, in contrast with the usual definitions of entropy as a functional of the distribution function [8], for the statistical theories the dynamical entropy is a linear functional of the distribution function. We show that when the dynamical entropy is evaluated at given equilibrium distribution functions, i.e. Boltzmann, Bose or Fermi distribution functions, it coincides with the usual entropy formulae coming from sum-of-states considerations. We can push this relationship further and show that for the statistical theories both expressions for the fluid entropy, the dynamical and the usual entropy, could be related in the following sense: if we write both expressions as integrals over the mass shell of certain functions in the cotangent space, then each integrand is the Legendre transform of the other. From this relationship we obtain that the convexity of the entropy functional implies the causality of the theory. Finally, we generalize this relation between the entropies to the pseudostatistical theories.

For completeness, we include appendix A with a short review of the relativistic dissipative theories of divergence form, and appendix B with a short motivation for the definitions presented in previous sections.

2. Statistical theories

The aim of this section is to define a particular class of dissipative divergence-type fluid theories, and to show its properties. The motivations for this definition are explained in appendix B, while a brief review of the dissipative divergence-type fluid theories, with a perfect fluid as the simplest example, is given in the appendix A.

For simplicity we start with non-dissipative theories. We say that a perfect fluid theory is of *statistical type* if its generating function $\chi(\zeta, \zeta^a)$ can be written in the following way,

$$\chi(\zeta, \zeta^a) = \int f(\zeta + p_a \zeta^a)\, \mathrm{d}\omega$$

where the integral is on the future mass shell $p_a p^a = -m^2$, assuming from now on $m = 1$; and $f(X)$ is a smooth scalar function which for large values of $X \equiv \zeta + p_a\zeta^a$ decays fast enough as to make the integral well defined and $\chi(\zeta, \zeta^a)$ smooth in all its variables. In order to avoid a loss of generality, we assume that the domain of definition of $f(X)$ is the entire real line and we leave for the examples further restrictions on this domain. We restrict ourselves to the case where ζ^a is timelike, so $p_a\zeta^a < 0$, for it is the scalar product of two future directed timelike vectors.

These theories have the property that the expressions for the particle-number current and the stress-energy tensors are given by, see also appendix B,

$$N_a = \int p_a f''(X)\, \mathrm{d}\omega$$

$$T_{ab} = \int p_a p_b f''(X)\, \mathrm{d}\omega$$

where $f''(X) = (\mathrm{d}^2 f)/(\mathrm{d}X^2)$. So, we interpret N_a and T_{ab} as the first two moments of a distribution function given by $f''(X)$. The simplest example is when $f(X) = k^2\, \mathrm{e}^{X/k}$, where k is the Boltzmann constant, namely Boltzmann gas.

An interesting property of this integral representation is that causality is easy to analyse. Indeed, following [6] (see appendix A), we say that a perfect fluid theory is causal if

$$t_a E^a = \tfrac{1}{2} t_a M^a_{AB} Z^A Z^B < 0$$

for all perturbations $Z^A = (\delta\zeta, \delta\zeta^a)$ and all future-directed timelike vectors t^a. For statistical perfect fluid theories, using the expression of χ in terms of mass shell integrals, this condition becomes:

$$t_a E^a = \frac{1}{2} \int (t_a p^a) f'''(X)(\delta\zeta + p_a \delta\zeta^a)^2\, \mathrm{d}\omega$$

for all perturbations $(\delta\zeta, \delta\zeta^a)$. Thus we can isolate from the set of all statistical theories a subclass of *manifestly causal theories*, namely those having† $f''' > 0$. Note that the Boltzmann gas is inside this subclass, which also includes Bose and Fermi gases, since for them we have

$$f''(X) = \frac{1}{\mathrm{e}^{-X/k} - \epsilon}$$

with $X = \zeta + p_a\zeta^a$ as above and $\epsilon = 0, 1$ or -1, for the Boltzmann, Bose or Fermi gases, respectively. Therefore, it is easy to see that for the three cases $f''' > 0$. Note that the domain of definition of $f(X)$ is different for Bose and Fermi gases; while the former is $X\epsilon(-\infty, 0)$, for the chemical potential, ζ, and $p_a\zeta^a$, are negative, in the latter is the whole real line, for ζ can take any real value. The generating functions χ for Bose and Fermi distribution functions are obtained by straightforward integration.

† Note that there are theories which are not manifestly causal, but nevertheless are causal.

We now turn to the problem of extending these theories outside equilibrium. We propose the following extension: given a statistical perfect fluid theory characterized by a function f, we define a *dissipative* divergence theory of statistical type by the generating function

$$\chi(\zeta, \zeta^a, \zeta^{ab}) = \int f(\zeta + p_a\zeta^a + p_a p_b \zeta^{ab})\, \mathrm{d}\omega. \tag{2.1}$$

This extension of the generating function χ is based on extending the domain of definition of f, from fluid states in the equilibrium submanifold ($\zeta^{ab} = 0$, see appendix A) to fluid states out of this submanifold ($\zeta^{ab} \neq 0$), in the way expressed in (2.1). Note that f could be defined only on the negative real line (for example, in the Bose distribution function) and, since to $p_a p_b \zeta^{ab}$ can take any real value, the above extension means that we have to analytically extend the domain of definition of f to all real numbers.

However, note that with this extension of the corresponding generating function χ is not well defined for arbitrary ζ^{ab}. We have a smooth extension only for ζ^{ab} which belong to $C^- = \{\zeta^{ab} | \zeta^{ab} = \zeta^{ba}, g_{ab}\zeta^{ab} = 0, l_a l_b \zeta^{ab} \leqslant 0, \forall l_a \text{ null}\}$ of maximal dimension. In particular, this extension of χ is discontinuous in $\zeta^{ab} = 0$. In the following lemma we explain the reason for this drawback, with a very simple example that contains this kind of behaviour.

Lemma 1. The function

$$F(c) \equiv \int_0^\infty f(-x + cx^2)x\, \mathrm{d}x$$

where f is any smooth, positive definite function, which is of compact support or decays faster than x^{-2}, is discontinuous at $c = 0$, having there a finite limiting value from the left and diverging from the right.

Proof. To prove that the limiting value from the left is finite just note that if c is negative, then the argument of f just grows in absolute value and so the decay assumptions on f imply convergence. To prove the statement about the the limiting value from the left, take a double step function (a small square), $sc(x)$, smaller than f, then we have

$$F(c) \geqslant G(c) \equiv \int_0^\infty sc(-x + cx^2)x\, \mathrm{d}x.$$

So it suffices to prove the divergence on the limit for the function $G(c)$. Assume for simplicity that $sc(x)$ is different from zero only in the interval $[-1, 1]$, and that there its value is 1. For positive values of c, the integral above has contributions from two intervals. If c is small, these intervals can be calculated up to first order in c obtaining $[0, 1 + c]$ and $[(1/c) - 1, (1/c) + 1]$. The contribution to the integral from the first one is finite for all values of c, while the second can be estimated using the mean value theorem to be bigger than $(2/c^2) + \frac{1}{4} + O(c)$ and so we see that it diverges for c tending to zero from positive values. □

This lemma gives a strong argument against the possibility of extending smoothly the definition of χ to values of ζ^{ab} such that $p_a p_b \zeta^{ab} > 0$, by simply extending the definition of f, no matter how smooth or how strong is the decay condition we impose on the extension†. However, of course, there exist a lot of smooth non-statistical extensions of the generating

† Near equilibrium the theory should behave, at least in some respects, like Eckart's theory. But in this theory there is a relation between ζ^{ab} and the derivatives of the flow velocity which renders totally unphysical any imposition on the sign of ζ^{ab}.

function χ to the presently forbidden values of ζ^{ab}. Any of these, essentially arbitrary, smooth extensions will be assumed to have been made in what follows; in particular, we require that extension in a neighbourhood of equilibrium, that is, in a neighborhood of the apex of the cone C^-. We note that the results on causality near equilibrium do not depend on the particular extension chosen outside C^-.

For these statistically extended dissipative fluid theories, a simple expression for the causality condition is also easy to obtain, namely

$$t_a E^a = \frac{1}{2} \int (t_a p^a) f'''(X)(\delta\zeta + p_a \delta\zeta^a + p_a p_b \delta\zeta^{ab})^2 \, \mathrm{d}\omega$$

but now with $X = \zeta + p_a \zeta^a + p_a p_b \zeta^{ab}$. So we have the following result.

Theorem 1. Let function $f : R \to R$ defining a statistical perfect fluid be C^3, and such that the equilibrium theory is well defined. If the equilibrium theory is manifestly causal, i.e. $f''' > 0$, then the extended dissipative theory is also causal in a neighbourhood of equilibrium.

Proof. The above expression shows causality for all values of $(\zeta, \zeta^a, \zeta^{ab})$ such that $\zeta^{ab} \in C^-$. Since the cone is of maximal dimension, partial derivatives along directions inside the cone suffice to determine completely the differentials of χ at the apex of the cone, that is at equilibrium. Thus the smooth extension outside the cone cannot change the causality properties of the equilibrium configuration. The result extends to a neighbourhood of equilibrium trivially by noticing that in our setting causality is a continuous property. □

Particular examples of this theorem are the dissipative extensions of Boltzmann, Bose and Fermi gases, given by $f'' = 1/(\mathrm{e}^{-X/k} - \epsilon)$, with $X = \zeta + p_a \zeta^a + p_a p_b \zeta^{ab}$, and $\epsilon = 0, 1$ or -1, for the Boltzmann, Bose or Fermi gases, respectively. Therefore, they are smooth and manifestly causal for all values of the parameters for which the integral expression converges, in and off equilibrium. This generalizes and simplifies our previous work [15], which in turn was a generalization of other results [13, 14].

3. Pseudostatistical theories

There is an even larger set of theories defined in terms of the integral of a certain function f on a future mass shell, for which it is straightforward to find sufficient conditions on f such that the resulting divergence theory is causal. We say that a dissipative fluid of divergence form is of *pseudostatistical* origin if its generating function χ can be written as

$$\chi(\zeta, \zeta^a, \zeta^{ab}) = \int f(\zeta + p_a p_b \zeta^{ab}, p_a \zeta^a) \, \mathrm{d}\omega$$

where the integral is on the future mass shell and $f = f(x, y)$ is a scalar function of two variables, $x \equiv \zeta + p_a p_b \zeta^{ab}$ and $y \equiv p_a \zeta^a$. We call them pseudostatistical because of the tensors N^a and A^{abc} can be thought of as coming from a distribution function† f_{xy}, while the stress-energy tensor T^{ab} can be thought of as coming from a *different* distribution function f_{yy}, in fact

$$N_a = \int p_a f_{xy} \, \mathrm{d}\omega$$

† Here the subindices indicate partial derivatives with respect to the argument signalled by the index.

$$T_{ab} = \int p_a p_b f_{yy}\, \mathrm{d}\omega$$

$$A_{abc} = \int p_a p_{<b} p_{c>} f_{xy}\, \mathrm{d}\omega$$

where the symbol $\langle\rangle$ means symmetrization and trace free, i.e. $p_{<b}p_{c>} = p_b p_c + g_{bc}/4$. The values of the variable ζ^a have the same restrictions as in section 2. Lemma 1 does not apply here, and there are cases like the Boltzmann where one can obtain a pseudostatistical extension for all values of ζ^{ab} starting from an equilibrium statistical theory.

The pseudostatistical theories are interesting since again the causality condition is simple and it is easy to impose a condition such that the theory is manifestly causal. Indeed, it is easy to see that in this case the causality condition is

$$\begin{aligned} t_a E^a = \frac{1}{2}\int (t_a p^a)[f_{xxy}(\delta\zeta)^2 + f_{yyy}(p_b\delta\zeta^b)^2 + f_{xxy}(p_{<d}p_{e>}\delta\zeta^{de})^2 \\ +2f_{xyy}\delta\zeta(p_b\delta\zeta^b) + 2f_{xxy}\delta\zeta(p_{<d}p_{e>}\delta\zeta^{de}) \\ +2f_{xyy}(p_b\delta\zeta^b)(p_{<d}p_{e>}\delta\zeta^{de})]\, \mathrm{d}\omega. \end{aligned}$$

Rearranging terms we have

$$\begin{aligned} t_a E^a = \frac{1}{2}\int (t_a p^a)[f_{xxy}(\delta\zeta + p_{<d}p_{e>}\delta\zeta^{de})^2 + f_{yyy}(p_b\delta\zeta^b)^2 \\ +2f_{xyy}(\delta\zeta + p_{<d}p_{e>}\delta\zeta^{de})(p_b\delta\zeta^b)]\, \mathrm{d}\omega \end{aligned}$$

and defining $\delta x = \delta\zeta + p_{<a}p_{b>}\delta\zeta^{ab}$ and $\delta y = p_a\delta\zeta^a$

$$t_a E^a = \frac{1}{2}\int (t_a p^a)[f_{xxy}(\delta x)^2 + f_{yyy}(\delta y)^2 + 2f_{xyy}\delta x\delta y]\, \mathrm{d}\omega.$$

Thus, if we assume $f_{xxy} \geqslant 0$, $f_{yyy} \geqslant 0$ and $(f_{xyy})^2 \leqslant f_{yyy}f_{xxy}$, we obtain

$$t_a E^a \leqslant \frac{1}{2}\int (t_a p^a)(\sqrt{f_{xxy}}|\delta x| - \sqrt{f_{yyy}}|\delta y|)^2\, \mathrm{d}\omega$$

and again isolate a subclass of dissipative theories (containing the one previously described) which are *manifestly causal*, namely those which satisfy

$$f_{xxy} \geqslant 0 \qquad f_{yyy} \geqslant 0 \qquad (f_{xyy})^2 \leqslant f_{yyy}f_{xxy}. \tag{3.1}$$

If we define a pseudostatistical perfect fluid as a theory given by $\chi = \int f(x, y)\, \mathrm{d}\omega$ with $x = \zeta$ and $y = p_a\zeta^a$, we thus have an obvious generalization of the previous theorem.

Theorem 2. Let function $f : R \times R^- \to R$ defining a perfect (equilibrium) pseudostatistical fluid be C^3, such that the equilibrium theory is well defined. If the equilibrium theory is manifestly causal, then the extended dissipative theory is also causal in a neighbourhood of equilibrium.

Remark. There are causal pseudostatistical theories whose f does not satisfy (3.1) so our condition is only sufficient.

4. Entropy

As we mention in the introduction and as can also be seen in appendix A, the requirement of an entropy law is a very strong condition which substantially constrains the kinematics and dynamics of the fluid theories. From this requirement, the fluid theory is determined by a freely given scalar generating function, χ (essentially an equation of state) and a, basically freely given, dissipation source tensor I^{ab}. The entropy current is explicitly obtained as a functional of the arbitrary generating function that defines each particular theory. This entropy is of dynamical origin and in principle it is not related to the entropy concept coming from information theory as applied to equilibrium configurations. In fact, the *entropy law* property can be divided into two conditions: the more stringent one is the existence of an extra vector field whose divergence, as a consequence of the other equations, is a local function of the dynamical field (and not of their derivatives), and a weaker one which requires this last function to be non-negative and to vanish at equilibrium. The first condition is the one which restricts the theory, gives its rigidity and determines completely the entropy functional, while the other condition in this picture just puts mild conditions on the dissipation-source tensor. We shall see in what follows how these conditions work for the natural extensions of statistical theories.

In the theories of divergence type the entropy is given by

$$\begin{aligned} S_a &= \frac{\partial \chi}{\partial \zeta^a} - \zeta N_a - \zeta^b T_{ab} - \zeta^{bc} A_{abc} \\ &= \chi_a - \zeta^A \frac{\partial \chi_a}{\partial \zeta^A} \end{aligned} \tag{4.1}$$

where $\zeta^A = (\zeta, \zeta^a, \zeta^{ab})$. For the naturally extended statistical theories this translates into

$$S_a = \int p_a (f' - P_A \zeta^A f'')\, \mathrm{d}\omega \tag{4.2}$$

where $P_A \equiv (1, p_a, p_a p_b)$, $P_A \zeta^A = X = \zeta + p_a \zeta^a + p_a p_b \zeta^{ab}$, and we recall that $m = 1$.

This expression for the entropy current is a *linear* functional of $f'(X)$ and so of the associated distribution function $F(X) = f''(X)$. We can relate this expression with the usual formula given in the form [8]

$$S_a = \int p_a \phi(F)\, \mathrm{d}\omega \tag{4.3}$$

where ϕ is an appropriate function of the distribution function F. The association is made by noting that functions $\phi(F)$ and $f'(X)$ are related by a Legendre transformation

$$\phi(F) = f'(X) - XF(X).$$

Therefore, from this association we infer that $\phi' = -X$ (where $\phi' = (\mathrm{d}\phi)/(\mathrm{d}F)$) and also that $\phi'' = -1/F'$ (where we used the previous definition of $F' = (\mathrm{d}F)/(\mathrm{d}X)$). The last relation is very important because it relates causality of the fluid theory ($f''' = F' > 0$) with the convexity of the entropy functional ($\phi'' < 0$). In other words, the convexity of the entropy functional implies the causality of the theory. This result is related with previous ones [16].

As an example of this relation, we can explicitly compute expression (4.2) for the following associated distribution function

$$f''(X) = F(X) = \frac{\eta}{\mathrm{e}^{-X/k} - \epsilon}$$

where η is a constant proportional to $1/h^3$ with h the Plank's constant. We recall that the fluid theory determined with the above associated distribution function is of statistical type, so it is defined only for those states in C^-. It is direct to see that

$$f' = -\frac{\eta k}{\epsilon} \left[\ln(\mathrm{e}^{-X/k} - \epsilon) + \frac{X}{k} \right].$$

If we define $\Delta = 1 + \epsilon F/\eta$, then we can write $\mathrm{e}^{-X/k} = (\eta\Delta)/F$; and

$$f' = \frac{\eta k}{\epsilon} \ln(\Delta).$$

So the entropy-current density functional S^a can be written as

$$\begin{aligned} S_a &= \int p_a(f' - Xf'')\,\mathrm{d}\omega \\ &= k \int p_a \left(\frac{\eta}{\epsilon} \ln(\Delta) + \ln\left(\frac{\eta\Delta}{F}\right) F \right) \mathrm{d}\omega \\ &= \, -k \int p_a \left(F \ln\left(\frac{F}{\eta}\right) - \frac{\eta}{\epsilon} \Delta \ln(\Delta) \right) \mathrm{d}\omega. \end{aligned}$$

The last expression corresponds to the usual entropy functional defined in relativistic kinetic theory whose equilibrium states correspond to the Bose or Fermi equilibrium distribution function if ϵ is 1 or -1, respectively. The case $\epsilon = 0$ corresponds to the Boltzmann distribution function, and is easy to see that S^a given by (4.1) has the usual form

$$S_a = -k \int p_a \left(F \ln\left(\frac{F}{\eta}\right) - F \right) \mathrm{d}\omega.$$

Thus the usual formulae for the entropy functionals is recuperated but, as we say above, they are only valid for states in C^-.

The interpretation of the statistical expression for the entropy (4.2), as the Legendre transform of the usual expression (4.3), can be generalized to pseudostatistical theories.

The expression for the pseudo-statistical entropy is

$$S_a = \int p_a(f_y - xf_{yx} - yf_{yy})\,\mathrm{d}\omega$$

where we used the conventions introduced in section 3. This expression can be thought of as a Legendre transform of an entropy function of the form

$$S_a = \int p_a \phi(F_1, F_2)\,\mathrm{d}\omega$$

where $F_1(x, y) \equiv f_{yx}(x, y)$ and $F_2(x, y) \equiv f_{yy}(x, y)$.

5. Conclusions

We have presented a variety of dissipative divergence-type theories with a statistical origin, in the sense that the tensors of the theory can be expressed as appropriate functions of the first three moments of a suitable distribution function. This represents a relation with kinetic theory, which is manifest in the integral expression for the resulting generating function χ. From this integral expression we could easily derive a simple condition on the associated distribution function for the resulting theory to be causal, even for some states far from momentary equilibrium states. This condition had been easily verified for the natural extensions to non-equilibrium states of equilibrium distribution functions associated with Boltzmann, Bose and Fermi statistics. The results obtained for these particular examples represent a great simplification of our previous work [15], which was a generalization of former works [13, 14].

The dynamical entropy defined in the divergence-type theories has not been related, in principle, to the entropy concept coming from a statistical theory as applied to equilibrium distribution functions. This dynamical entropy is thought to be just a vector field (constructed from the fluid fields) whose divergence is a pointwise function in the fields and its definition does not use anything about equilibrium configurations. This expression has a surprising form, for in the theories here considered it is linear on the associated distribution function. A remarkable fact of this dynamical entropy is that, for momentary equilibrium configurations corresponding to the distribution functions associated with the usual Boltzmann, Bose and Fermi statistics, it takes the familiar form one encounters in statistical mechanics when applied to equilibrium configurations. We push the relation between the dynamical and statistical or thermodynamical entropy to all statistical theories, in the sense that both expressions, represented as integrals over the mass shell, are related by a Legendre transform of their integrands. From this relation we obtain that the convexity of the entropy functional implies the causality of the theory.

There exists a serious limitation for this integral representation of the statistical theories, as has been shown with lemma 1. This lemma says that we cannot extend smoothly the definition of χ, to values of ζ^{ab} such that $\zeta^{ab} \notin C^-$, simply by extending the definition of f to such values. In other words, the statistical interpretation of χ cannot be extended. We can, of course, extend smoothly the definition of χ to the forbidden values of ζ^{ab} in an almost arbitrary way. All those extensions will be, by continuity, causal in a neighbourhood of the region where $p_a p_b \zeta^{ab}$ is non-negative, in particular in a neighbourhood of equilibrium. Does the limitation in the integral representation have any physical interpretation? The condition $l_a l_b \zeta^{ab} \leqslant 0, \forall l^a$ null, seems to be unphysical if we look near equilibrium, for there the theory should resemble Eckart [4], and it is known that there does not exist any restricting condition on the dissipative variable ζ^{ab} in that theory. Therefore, these properties of statistical theories let us propose the following conjecture: the fluid approximation, that is the approximate description of a distribution function's evolution via the transport equation, based on a finite number of macroscopic variables ($\zeta^A = (\zeta, \zeta^a, \zeta^{ab})$ for the Grad 14-parameter approximation), together with the requirement of a dynamical entropy law, is physically inconsistent. While this approximation seems to work very well at equilibrium, it might be that it fails completely away from equilibrium†. For an appropriate fluid description considering dissipation only with one particle distribution function an infinite set of field variables are needed. In this case a natural extension seems to be possible, provided the distribution function is suitably extended (if it is necessary) for positive values of its argument. This solution is of the type *add more dynamical fields*, in order to get some sort of cut-off. Another type of possible extension is to abandon the direct relation between distribution moments and dynamical variables outside equilibrium, but still retain the flavour of a statistical theory in the sense of the representation as integral over mass shells. This is easily implemented by changing the argument of the distribution function in a nonlinear way, for instance, $x = (\zeta + p_a\zeta^a + p_a p_b\zeta^{ab} + \lambda(p_a p_b\zeta^{ab})^2)$. This solution is of the type *add a parameter*, in order to get a nonlinear cut-off. Both proposals sound interesting and perhaps the solution lies in between. It is also possible that the problem arises when one pretends to describe a dissipative fluid with one particle distribution function, neglecting the effects of correlations among the microscopic particles that constitute the fluid.

It has been possible to extend the proof of causality given for the statistical theories to a bigger set of divergence theories. For this set, the three tensor fields, N^a, T^{ab} and A^{abc}, are also averages over a future mass shell, but for different (although related) distribution functions. We do not have any application for this larger class, but believe it should be of relevance for describing some physical phenomena. In this case extensions seem to

† It is not clear in this context what we mean by expressions like *near equilibrium* and its relation with, for instance, the *the size of* ζ^{ab}.

be possible for all values of the non-equilibrium variables, but such extensions in general are not causal for all such values, and so do not seem to be very physical. Are there special extensions which from a manifestly causal equilibrium statistical theory yield a causal pseudostatistical non-equilibrium theory?

Appendix A. Divergence-type fluid theories

Following [6, 11, 12], we assume that a fluid state is characterized by a finite collection of spacetime tensor fields, $\zeta^A = (\zeta, \zeta^a, \zeta^\alpha)$, where upper case indices denote the entire set of spacetime indices, and fields with Greek indices denote spacetime fields different for a scalar and a vector fields. We define a *divergence-type fluid theory* by the condition that these fluid fields are subject to the following first-order system of partial differential equations,

$$\nabla_a N^a = 0 \tag{A.1}$$

$$\nabla_a T^{ab} = 0 \tag{A.2}$$

$$\nabla_a A^{a\alpha} = I^\alpha \tag{A.3}$$

where N^a is the particle-number current, T^{ab} is the stress-energy tensor, while $A^{a\alpha}$ and I^α represent a collection of tensors, functions of the fluid fields and the spacetime metric, necessary to obtain a consistent system of equations. We also assume the existence of a dynamical entropy law, that is, there exists a vector field, called the entropy current, with the property that its divergence as consequence of the dynamical equations contains no derivatives of the dynamical fields and furthermore is positive semidefinite. This non-trivial condition implies that there exists a *single scalar* function of the dynamical fields χ, called the generating function, that determines *all* the principal parts of the dynamical system of equations of the theory. Therefore, it is possible to find the following expressions for tensors N^a, T^{ab} and $A^{a\alpha}$ [6, 11, 12],

$$N_a = \frac{\partial^2 \chi}{\partial \zeta^a \partial \zeta} \qquad T_{ab} = \frac{\partial^2 \chi}{\partial \zeta^a \partial \zeta^b} \qquad A_{a\alpha} = \frac{\partial^2 \chi}{\partial \zeta^a \partial \zeta^\alpha}$$

and the entropy current is given by $S_a = (\partial \chi)/(\partial \zeta^a) - \zeta N_a - \zeta^b T_{ab} - \zeta^\alpha A_{a\alpha}$, while its divergence is $\sigma = -\zeta^\alpha I_\alpha$.

Finally, it will be useful to rewrite the dynamical system of equations of a divergence fluid theory (A.1)–(A.3) as a first-order system of partial differential equations [6]

$$M^a_{AB} \nabla_a \zeta^B = I_A \tag{A.4}$$

where M^a_{AB} is expressed in terms of the generating function χ as follows:

$$M^a_{AB} = \frac{\partial^3 \chi}{\partial \zeta_a \partial \zeta^A \partial \zeta^B}.$$

Note that $M^a_{AB} = M^a_{(AB)}$ since partial derivatives commute; and that we have defined tensors $I_A = (0, 0, I_\alpha)$, where the two zeros represent the conservation equations for the particle-number current and the stress-energy tensor.

We say that system (A.4) is symmetric hyperbolic if $M^a_{AB} = M^a_{(AB)}$ and there exists a timelike-future directed t^a such that $N_{AB} \equiv -t_a M^a_{AB}$ is positive definite. This is a sufficient condition to have a well-posed initial value formulation. We say that system (A.4) is causal if it is symmetric hyperbolic for all future-directed timelike vectors t^a. This is a stronger condition required on physical grounds and means that the characteristic surfaces of system (A.4) are inside the null cone given by the spacetime metric. Physically this means that no information generated from the fluid can be propagated faster than light. The causality condition can be written in an equivalent form saying that the 4-vector

$$E^a \equiv \tfrac{1}{2} M^a_{AB} Z^A Z^B$$

is a future-directed timelike vector, for all non-vanishing Z^A.

Finally, we analyse the structure of the equilibrium states. Following [6, 7] we say that a fluid state ζ^A, a solution of the dynamical system of equations, is a *strictly equilibrium state* if its time reverse is also a solution. We denote a strictly equilibrium state by ζ_0^A, and we also put a subindex zero to any tensor evaluated at an equilibrium state. This definition implies that $I_{0A} = 0$ and as a consequence we have the fact that the production of entropy vanishes. If we also require that the theory be generic in the sense that the first variations of tensor I_A around an equilibrium state be an arbitrary tensor, we can conclude that $\zeta_0^\alpha = 0$ (see [6]) and so at strict equilibrium we have $\zeta_0^A = (\zeta_0, \zeta_0^a, 0)$. Finally, another genericity requirement, but now on the principal part of the equations, implies that the presence of dissipation affects all the fields of the theory, in the sense of not allowing for a decoupled set of fields with its own evolution not being driven, even indirectly by dissipation. This genericity condition is formally defined in [6, 17] and implies that ζ_0 is constant and ζ_0^a is a Killing vector. For these strictly equilibrium states the norm of ζ^a, that is $\mu \equiv \sqrt{-\zeta^a \zeta_a}$, can be associated with one over the temperature of the fluid, and its direction $u^a \equiv \zeta^a/\mu$ with the 4-velocity of the fluid. The variable ζ can be associated with a chemical potential per unit temperature of the fluid. Following [4] we also define a *momentary equilibrium state* by only the condition that $\zeta^\alpha = 0$. We call them momentary, because if this condition holds for a certain moment of time then there is no guarantee that it will hold in subsequent times. We call them equilibrium states, because at them the entropy production vanishes.

The simplest physical system described by these divergence-type theories is a perfect fluid, corresponding to a generating function χ depending only on the non-dissipative variables ζ and ζ^a. The particle-number current and the stress-energy tensor are given by the usual expressions

$$N^a = -\chi_{,\zeta\mu} u^a \qquad T^{ab} = \chi_{,\mu\mu} u^a u^b - \frac{\chi_{,\mu}}{\mu} q^{ab}$$

where $q^{ab} = g^{ab} + u^a u^b$, $\zeta^a = \mu u^a$ with $u_a u^a = -1$, and subindices $\zeta\mu$ in χ indicates differentiation. So we have the particle number of the fluid $n = -\chi_{,\zeta\mu}$, the energy density $\rho = \chi_{,\mu\mu}$ and pressure $p = -\chi_{,\mu}/\mu$ in terms of χ. It can be seen that in this theory μ is the inverse of the temperature of the fluid and ζ is a chemical potential per unit temperature. The choice $\chi = \chi(\zeta, \zeta^a)$ implies that the tensors $A^{a\alpha}$ are identically zero, and this is consistent with the choice $I^\alpha = 0$, for the dissipation-source tensors. In these theories the expression for the entropy current reduces to the usual one, $S^a = nsu^a$, and it satisfies $\nabla_a S^a = 0$, where $s = (\rho + p)/(nT) - \zeta$ is the entropy per volume and per particle.

Appendix B. Motivations from kinetic theory

In this appendix we present the motivations arising from the kinetic theory of gases for the definitions introduced earlier. A detailed presentation of kinetic theory in general relativity is given in [18].

A distribution of identical particles which interacts via short-range forces, idealized as point collisions, in a given fixed spacetime, is described with a distribution function F and a collision term $\mathcal{C}$, both being functions on the phase space. Given an expression for the collision term, in general a functional of F, the transport equation determines the dynamics of F. The corresponding initial value formulation for F is complicated, so alternative methods are used to obtain information about the system, such as approximation methods or truncations that yield macroscopic equations, like the perfect fluid equations. These methods were known long ago [9, 10, 19–21] and they are based on the expectation that the dynamics of the Boltzmann equation on the cotangent space can be well approximated by the dynamics of a *finite* number of spacetime fields, the macroscopic variables. The dynamical equations for these variables are the equations for some of the associated moments of the distribution function.

For example, in the perfect fluid approximation we assume $F = F(\zeta(q), p_a\zeta^a(q))$; therefore, to obtain a closed system of equations for the macroscopic variables (ζ, ζ^a) it is necessary to consider only the equations for the first two moments $\nabla_a N^a = 0$ and $\nabla_a T^{ab} = 0$, which are the conservation of the particle-number current, N_a, and the stress-energy tensor of the fluid, T_{ab}. It is easy to check by straightforward integration on the mass shell of a distribution function of the form $F(\zeta(q), p_a\zeta^a(q))$ that the first two moments correspond to the perfect fluid particle-number and stress-energy tensor, respectively, that is they are characterized by the particle number n, the energy density ρ and the presure p, of the fluid as measured by an observer moving with this fluid, that is with 4-velocity u^a. These functions n, ρ and p depend on ζ and μ with $\zeta^a = \mu u^a$. Up to this point, n, ρ and p are three arbitrary functions of two scalar functions ζ and μ without clear meaning. From the first and second laws of thermodynamics we know that there exists a function $S(\rho, n)$, the entropy per volume, such that

$$\nabla_a S = \mu \nabla_a \rho - \zeta \nabla_a n$$

where we have identified the function μ with one over the temperature and the function ζ with a chemical potential per unit temperature. This implies a restriction on the function p such that $(\partial p)/(\partial \zeta) = n/\mu$ and $(\partial p)/(\partial \mu) = -(\rho + p)\mu$. These conditions imply that the distribution function must satisfy $F(\zeta, p_a\zeta^a) = F(\zeta + p_a\zeta^a)$. Therefore, only distribution functions depending on $X = \zeta + p_a\zeta^a$ generate a perfect fluid with an entropy law. This condition is the motivation to define the perfect fluids of statistical type.

It is possible to consider a further approximation, which is known as the 14-parameter Grad approximation, assuming that $F = F(\zeta(q), p_a\zeta^a(q), p_a p_b \zeta^{ab}(q))$, where the spacetime tensor field ζ^{ab} is, without loss of generality, symmetric and trace free (the last condition is because the trace part is the scalar field ζ). To obtain a closed system of equations for the macroscopic variables $(\zeta, \zeta^a, \zeta^{ab})$, it is necessary to only consider the equations for the first three moments

$$\nabla_a N^a = 0 \tag{B.1}$$

$$\nabla_a T^{ab} = 0 \tag{B.2}$$

$$\nabla_a J^{a\langle bc\rangle} = \boldsymbol{I}^{\langle bc\rangle} \tag{B.3}$$

where the symbol $\langle\rangle$ means symmetrization and trace free, and we have assumed the former identification of the first two moments with particle-number and stress-energy tensor, respectively. The system (B.1)–(B.3) is of the form (A.1)–(A.3) for the special case of dissipative variables $\zeta^\alpha = \zeta^{ab}$ with ζ^{ab} symmetric and trace free. Therefore, if we restrict our study to a particular class of systems described by equations (B.1)–(B.3) such that they contain a *dynamical entropy law*, then there exists a single scalar generating function $\chi(\zeta, \zeta^a, \zeta^{ab})$, such that all tensors appearing in the principal part of the dynamical equation can be written in terms of χ.

In appendix A it was assumed that the dynamical equations of the dissipative fluid theory consist in the two conservations laws corresponding to particle-number and stress-energy tensors, and other equations which could also be written as a total divergence of an appropriate tensor, without further restrictions on it. With only these assumptions, the generating function could be any scalar function of fluid variables, without any restriction on it. However, in the particular system of dynamical equations that we are considering here, (B.1)–(B.3), the third equation is the divergence of a third moment of a distribution function. This implies that it is a totally symmetric tensor, and its trace in any pair of indices is proportional to the particle-number current. It can be checked that this extra condition implies a restriction on the generating function χ that can be expressed saying that it must be a solution of the following system of partial differential equations

$$\frac{\partial^2\chi}{\partial\zeta^{[a}\partial\zeta^{b]c}} - \frac{1}{4}\frac{\partial^2\chi}{\partial\zeta\partial\zeta^{[a}}g_{b]c} = 0 \tag{B.4}$$

where symbol [] means antisymmetrization. Therefore, this equation expresses in a formal way the statistical origin of the dissipative divergence-type theory (B.1)–(B.3).

We obtained solutions of (B.4) with a natural anzatz, also inspired in kinetic theory. We proposed a solution of the form

$$\chi(\zeta, \zeta^a, \zeta^{ab}) = \int \hat{f}(x, y, z)\,\mathrm{d}\omega$$

where $x = \zeta$, $y = p_a\zeta^a$ and $z = p_a p_b \zeta^{ab}$, the integral is on the future mass shell $p_a p^a = -1$, with $\hat{f} : R^3 \to R$ and for x, y, z large it is assumed that $\hat{f}$ behaves in such a way that the integral converges. With this assumption it is direct to write equation (B.4) for function $\hat{f}$ as

$$0 = \int [p_{[a}(p_{b]}p_c + \tfrac{1}{4}g_{b]c})\hat{f}_{yz} - \tfrac{1}{4}p_{[a}g_{b]c}\hat{f}_{yx}]\,\mathrm{d}\omega = -\frac{1}{4}\int p_{[a}g_{b]c}\hat{f}_{y(x-z)}\,\mathrm{d}\omega$$

where subindices in $\hat{f}$ indicate differentiation and we have used the fact that derivatives respect to ζ^{ab} keep the trace free property. The last equation implies that a family of solutions is obtained by the condition

$$\hat{f}(x, y, z) = f(x + z, y) + \tilde{f}(x, z).$$

The second term in the equation above does not affect the dynamics of the fluid because the function with physical meaning is $\chi_a = (\partial\chi)/(\partial\zeta^a)$, as can be seen in appendix A. This function is related only to $\hat{f}_y$, so we can discard $\tilde{f}$. Therefore, we have obtained the following family of solutions

$$\chi(\zeta, \zeta^a, \zeta^{ab}) = \int f(x + z, y)\,\mathrm{d}\omega$$

which we called pseudostatistical theories in section 3, which includes the statistical theories defined in section 2.

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